Math 2X03 - Homework 6

Due: June 14, 2016

Chapters Covered: Chapter 16.6, 16.7, 16.8, 16.9

1. Find the area of the following surfaces:

   (a) (Chapter 16.6 # 45) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

   (b) (Chapter 16.6 #49) The surface with parametric equations $x = u^2$, $y = uv$ and $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2$.

2. (Chapter 16.7 # 17) Evalute the surface integrals

$$\iint_S (x^2 z + y^2 z)\, dS,$$

   where S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$.

3. (Chapter 16.7 #30) Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 5\mathbf{k}$ across the surface S, where S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$, and the planes $y = 0$ and $x + y = 2$.

4. We showed in class that in the case of surface S (with upward orientation of S) is given by a graph $z = g(x, y)$ and a vector field $\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$, the flux of $\mathbf{F}$ across a surface S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

   Find a similar formula for the case where S is given by $y = h(x, z)$ and $\hat{\mathbf{n}}$ is the unit normal that points towards the **left**.

5. (Chapter 16.7 #48) The termperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball. (See Example 6 in Chapter 16.7)

6. Use Stokes' Therem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is **oriented counterclockwise as viewed from above**.

   (a) $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$, where C is the boundary of part of the plane $3x + 2y + z = 1$ in the first octant.

   (b) $\mathbf{F}(x, y, z) = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$, where C is the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \ge 0$.

7. (Chapter 16.8 #16) Let C be a simple closed curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\, dx - 2x\, dy + 3y\, dz$$

   depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

8. Use Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S:

   (a) (Chapter 16.9 # 10) $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + zx\mathbf{k}$, where S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

   $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

   where a, b, c are psitive numbers.

   (b) (Chapter 16.9 #13) $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disc $x^2 + y^2 \leq 1$ in the xy-plane.

9. Evaluate the total flux of the vector field $\mathbf{F} = (x^3 + 2yz, y^3 + 2xz, z^3 + 2xy)$ across the lower hemisphere $S = \{x^2 + y^2 + z^2 = 4, z \leq 0\}$ with respect to the downward pointing unit normal.

10. (a) Compute the total outward flux of the vector field

   $$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

   across the surface $(x - 2)^2 + 2(y - 5)^2 + 3(z - 3)^2 = 1$ (an ellipsoid).

   (b) For the vector field $\mathbf{F}$ in part (a) above, can one find a vector field $\mathbf{G}$, with the same domain as $\mathbf{F}$, so that $\mathbf{F} = \nabla \times \mathbf{G}$? Give a full justification for your answer.

11. (a) Compute the area enclosed by the loop in the curve $\vec{c}(t) = (t^2, \frac{t^3}{3} - t)$ where $-\sqrt{3} \leq t \leq \sqrt{3}$ by using Green's theorem.

   (b) Compute the total outward flux of the vector field

   $$\mathbf{F} = \big(x - yz + 1,\ y + xz, z + yx^2\big)$$

   across the boundary of the solid region $\{z^2 \geq x^2 + y^2,\ 1 \leq z \leq 2\}$.

12. Let S denote the surface given by $y = 10 - x^2 - z^2, y \geq 1$ oriented with the rightward pointing normal. Let

   $$\mathbf{F} = (2xyz + 3z, e^x \cos(yz) - e^{-x} \sin(yz), -3xy^2).$$

   Evaluate the integral

   $$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$